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A semi closed-form analytic pricing formula for call options in a hybrid Heston–Hull–White model

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1 Introduction

We consider the valuation of European call options in a general asset pricing model

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW_t^1 \,, \\ dv_t &= \kappa(\eta - v_t)dt + \lambda\sqrt{v_t}\; dW_t^2 \,, \\ dr_t &= (\theta(t) - a r_t)dt + \sigma dW_t^3 \end{cases} \tag{1.1}$$

for $0 \le t \le T$ with T the maturity of the option. Here S_t, v_t, r_t denote random variables that represent the stock price, its variance and the interest rate, respectively, at time $t \ge 0$. The model (1.1) constitutes an extension of the well-known Black–Scholes model [3] where the volatility and the interest rate both evolve randomly over time.

The process for the variance v_t has been proposed by Heston [5]. The process for the interest rate r_t was formulated by Hull & White [6] and forms a generalization of the Vasicek model [8]. The quantities κ, η, λ, a, σ are positive real constants. Further $\theta(t)$ is a deterministic, continuous, positive function of time which can be chosen as to match the current term structure of interest rates. Finally, W_t^1, W_t^2, W_t^3 denote Brownian motions. We assume that the process W_t^3 is independent from W_t^1 and W_t^2. The two Brownian motions W_t^1, W_t^2 are allowed to be correlated; their correlation is denoted by $\rho \in [-1, 1]$.

The purpose of this note is to derive an analytic pricing formula in semi closed-form for European call options under the asset pricing model (1.1). The availability of such a pricing formula is particularly useful in a calibration procedure. In practice, option pricing models are calibrated to a large number of market-observed call option prices. It is important that such a parameter estimation procedure is fast. Therefore a (near) closed-form call option pricing formula is very desirable.

Our analysis in this note follows the lines of Heston [5]. The formula that we obtain forms a direct extension of Heston's pricing formula for call options, which can quickly be evaluated.

2 A semi closed-form analytic formula for call option prices

Let $C(t, s, v, r)$ denote the price of a European call option at time $t \in [0, T]$ given that at this time the asset price equals s, its variance equals v and the interest rate equals r.

From standard no-arbitrage arguments it follows that C satisfies the parabolic partial differential equation (PDE)

$$\begin{aligned} 0 \;=\; & \frac{\partial C}{\partial t} + \tfrac{1}{2}s^2 v \frac{\partial^2 C}{\partial s^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 C}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial r^2} + \rho\lambda s v \frac{\partial^2 C}{\partial s \partial v} \\ & + r s \frac{\partial C}{\partial s} + \kappa(\eta - v)\frac{\partial C}{\partial v} + (\theta(t) - a r)\frac{\partial C}{\partial r} - rC \end{aligned} \tag{2.1}$$

for $0 \le t < T$, $s > 0$, $v > 0$, $-\infty < r < \infty$. This PDE can be viewed as a time-dependent advection-diffusion-reaction equation on an unbounded, threedimensional spatial domain. The payoff of a call option yields the terminal condition

$$C(T, s, v, r) = \max(0, s - K), \tag{2.2}$$

where $K > 0$ is the strike price of the call option. Further, a boundary condition at $s = 0$ holds,

$$C(t, 0, v, r) = 0 \quad (0 \le t < T). \tag{2.3}$$

We note that at $v = 0$ no condition is specified.

It is convenient to first apply a change of variables. Define

$$\hat{C}(t, x, v, r) = C(t, e^x, v, r). \tag{2.4}$$

Then $\hat{C}$ satisfies the PDE

$$\begin{aligned} 0 &= \frac{\partial \hat{C}}{\partial t} + \tfrac{1}{2} v \frac{\partial^2 \hat{C}}{\partial x^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 \hat{C}}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 \hat{C}}{\partial r^2} + \rho\lambda v \frac{\partial^2 \hat{C}}{\partial x \partial v} \\ &\quad + (r - \tfrac{1}{2} v)\frac{\partial \hat{C}}{\partial x} + \kappa(\eta - v)\frac{\partial \hat{C}}{\partial v} + (\theta(t) - ar)\frac{\partial \hat{C}}{\partial r} - r\hat{C} \end{aligned} \tag{2.5}$$

for $0 \le t < T$ on the spatial domain $(x, v, r) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}$ with terminal condition

$$\hat{C}(T, x, v, r) = \max(0, e^x - K). \tag{2.6}$$

As in [5], we guess a solution of the form similar to the Black–Scholes formula:

$$\hat{C}(t, x, v, r) = e^x P_1(t, x, v, r) - KB(t, r)P_2(t, x, v, r). \tag{2.7}$$

Here $B(t, r)$ denotes the time-t value of a zero-coupon bond that pays off 1 at maturity, given that at time t the short rate equals r. It satisfies the PDE

$$0 = \frac{\partial B}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 B}{\partial r^2} + (\theta(t) - ar)\frac{\partial B}{\partial r} - rB \tag{2.8}$$

for $0 \le t < T,\ r \in \mathbb{R}$ and a semi closed-form solution is given by

$$B(t, r) = e^{b(t,r)}, \tag{2.9a}$$

$$\begin{aligned} b(t, r) &= -\frac{r}{a}\left(1 - e^{-a(T-t)}\right) - \frac{1}{a}\int_t^T \theta(s)\left(1 - e^{-a(T-s)}\right) ds \\ &\quad + \frac{\sigma^2}{2a^2}\left(T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right). \end{aligned} \tag{2.9b}$$

By linearity, the guess (2.7) satisfies the PDE (2.5) if its two constituent terms satisfy (2.5). As such, P_1 satisfies the PDE

$$\begin{aligned} 0 &= \frac{\partial P_1}{\partial t} + \tfrac{1}{2} v \frac{\partial^2 P_1}{\partial x^2} + \tfrac{1}{2}\lambda^2 v \frac{\partial^2 P_1}{\partial v^2} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 P_1}{\partial r^2} + \rho\lambda v \frac{\partial^2 P_1}{\partial x \partial v} + \\ &\quad (r + \tfrac{1}{2} v)\frac{\partial P_1}{\partial x} + [\kappa(\eta - v) + \rho\lambda v]\frac{\partial P_1}{\partial v} + (\theta(t) - ar)\frac{\partial P_1}{\partial r}, \end{aligned} \tag{2.10}$$

and by invoking (2.8), P_2 satisfies

$$\begin{aligned} 0 \quad = \quad & \frac{\partial P_2}{\partial t} + \tfrac{1}{2}v\frac{\partial^2 P_2}{\partial x^2} + \tfrac{1}{2}\lambda^2 v\frac{\partial^2 P_2}{\partial v^2} + \tfrac{1}{2}\sigma^2\frac{\partial^2 P_2}{\partial r^2} + \rho\lambda v\frac{\partial^2 P_2}{\partial x \partial v} + \\ & (r - \tfrac{1}{2}v)\frac{\partial P_2}{\partial x} + \kappa(\eta - v)\frac{\partial P_2}{\partial v} + \left[\theta(t) - ar + \sigma^2\frac{\partial b}{\partial r}\right]\frac{\partial P_2}{\partial r}\,. \end{aligned} \tag{2.11}$$

Further, (2.6) yields for the PDEs (2.10), (2.11) the terminal conditions

$$P_j(T, x, v, r) = 1 \quad (x > \ln K) \quad , \quad P_j(T, x, v, r) = 0 \quad (x < \ln K) \tag{2.12}$$

for $j = 1, 2$, respectively.

From the undiscounted, multidimensional version of the Feynman–Kac Theorem (cf. [7]) it follows that the solutions P_1, P_2 to (2.10), (2.11) with (2.12) can be written as expectations of the indicator function corresponding to (2.12), and thus can be regarded as probabilities[1]. We next derive semi closed-form formulas for P_1 and P_2 by solving for their characteristic functions. From these characteristic functions the probabilities P_1, P_2 can be retrieved with the inversion theorem (cf. [4, 5]):

$$P_j(t, x, v, r) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\ln K} f_j(t, x, v, r; u)}{iu}\right] du \quad \text{for } j = 1, 2 \tag{2.13}$$

where $i^2 = -1$.

The Feynman–Kac theorem directly yields that the functions f_1, f_2 satisfy the same PDEs (2.10), (2.11), respectively, but with the terminal condition

$$f_j(T, x, v, r; u) = e^{iux}. \tag{2.14}$$

For f_1 we guess a solution of the form (cf. [5])

$$f_1(t, x, v, r; u) = \exp[F_1(t; u) + G_1(t; u)v + H_1(t; u)r + iux]. \tag{2.15}$$

Substituting this into the PDE (2.10), it follows by perusal of the coefficients of v, r and 1 that (2.15) is a solution if the functions F_1, G_1, H_1 satisfy the system of ordinary differential equations (ODEs)

$$F_1'(t) + \kappa\eta G_1(t) + \theta(t)H_1(t) + \tfrac{1}{2}\sigma^2 H_1(t)^2 = 0\,, \tag{2.16a}$$

$$G_1'(t) + \tfrac{1}{2}ui - \tfrac{1}{2}u^2 + (\rho\lambda ui + \rho\lambda - \kappa)G_1(t) + \tfrac{1}{2}\lambda^2 G_1(t)^2 = 0\,, \tag{2.16b}$$

$$H_1'(t) + ui - aH_1(t) = 0\,, \tag{2.16c}$$

with the terminal condition $F_1(T) = G_1(T) = H_1(T) = 0$.

For f_2 we guess a solution of the form (cf. [2, 5])

$$f_2(t, x, v, r; u) = \exp[F_2(t; u) + G_2(t; u)v + H_2(t; u)r + iux - b(t, r)]. \tag{2.17}$$

[1] We omit the details, which are completely analogous to those explained in [5].

Substituting this into the PDE (2.11) and using (2.8),(2.9), it follows analogously as above that (2.17) is a solution if the functions F_2, G_2, H_2 satisfy the system of ODEs

$$\begin{aligned} &F_2'(t) + \kappa\eta G_2(t) + \theta(t) H_2(t) + \tfrac{1}{2}\sigma^2 H_2(t)^2 = 0\,, && \text{(2.18a)}\\ &G_2'(t) - \tfrac{1}{2}ui - \tfrac{1}{2}u^2 + (\rho\lambda ui - \kappa) G_2(t) + \tfrac{1}{2}\lambda^2 G_2(t)^2 = 0\,, && \text{(2.18b)}\\ &H_2'(t) + ui - aH_2(t) - 1 = 0\,, && \text{(2.18c)} \end{aligned}$$

with the terminal condition $F_2(T) = G_2(T) = H_2(T) = 0$.

The equations (2.16c), (2.18c) are easy to solve. Let $\delta_1 = 0$, $\delta_2 = 1$. Then

$$H_j(t;u) = \frac{ui - \delta_j}{a}\left(1 - e^{-a(T-t)}\right) \qquad \text{for } j = 1, 2. \tag{2.19}$$

The equations (2.16b), (2.18b) are identical[2] to the first line of equation (A7) in [5] and closed-form solutions were obtained in loc. cit. For completeness, we include these formulas here. Let

$$\alpha = \kappa\eta \quad , \quad \beta_1 = \kappa - \rho\lambda \quad , \quad \beta_2 = \kappa \quad , \quad \gamma_1 = \frac{1}{2} \quad , \quad \gamma_2 = -\frac{1}{2}$$

and for $j = 1, 2$

$$d_j = \sqrt{(\beta_j - \rho\lambda ui)^2 - \lambda^2(2\gamma_j ui - u^2)} \quad , \quad g_j = \frac{\beta_j - \rho\lambda ui + d_j}{\beta_j - \rho\lambda ui - d_j}\,.$$

Then the solutions to (2.16b), (2.18b) are given by

$$G_j(t;u) = \frac{\beta_j - \rho\lambda ui + d_j}{\lambda^2}\left[\frac{1 - e^{d_j(T-t)}}{1 - g_j e^{d_j(T-t)}}\right] \qquad \text{for } j = 1, 2. \tag{2.20}$$

The equations (2.16a), (2.18a) can finally be solved by integration. Using the result from [5] for the integral of G_j, it follows that

$$\begin{aligned} F_j(t;u) \quad = \quad & \frac{\alpha}{\lambda^2}\left\{(\beta_j - \rho\lambda ui + d_j)(T-t) - 2\ln\left[\frac{1 - g_j e^{d_j(T-t)}}{1 - g_j}\right]\right\} \\ & + \frac{ui - \delta_j}{a}\int_t^T \theta(s)\left(1 - e^{-a(T-s)}\right) ds \\ & + \frac{\sigma^2}{2}\left(\frac{ui - \delta_j}{a}\right)^2\left(T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right) \end{aligned} \tag{2.21}$$

for $j = 1, 2$. Of course, for many functions θ the integral in (2.21) may be explicitly computed.

[2] With the proper change of notation and removing a typo in [5].

The formulas (2.4), (2.7), (2.9), (2.13), (2.15), (2.17), (2.19), (2.20), (2.21) together constitute the semi closed-form pricing formula for European call options under the asset pricing model (1.1). This pricing formula is easily seen to be a proper extension of Heston's formula, upon considering $\theta(t) \equiv ar_0$ and $\sigma = 0$.

If the integrals in (2.9b), (2.21) involving $\theta(s)$ can be explicitly computed, the pricing formula consists of two single integrals over u, see (2.13). Otherwise, one has an additional single integral over s,

$$\int_t^T \theta(s) \left(1 - e^{-a(T-s)}\right) ds\,.$$

Note the useful property that the latter integral does not depend on u. In all cases, the pricing formula can be quickly approximated to any accuracy with a suitable numerical integration method. For a discussion of some computational issues relevant to the pricing formula, we refer to the paper [1] on the Heston formula.

Finally, we remark that two issues are not addressed in this note, namely whether the solution obtained above is unique and whether it satisfies the condition (2.3). These two issues are left for future research. We note that it is plausible that the probability $P_2(t, x, v, r)$ in (2.7) vanishes as $x \to -\infty$, and therefore that (2.3) holds. But, this requires a careful analysis of course.

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